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The w-integral closure of integral domains

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Abstract

Let D be an integral domain with quotient field K, $\bar{D}$ the integral closure of D, X an indeterminate over D, and $N_v = \{f \in D[X] \mid (A_f)_v = D\}$. Let w be the $*$-operation on D defined by $I_w = \{x \in K \mid$ there is a finitely generated ideal A such that $A^{-1} = D$ and $xA \subseteq I\}$, and let $D^w = \{u \in K \mid uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of $D\}$. Then D^w, called the w-integral closure of D, is an integrally closed overring of D. In this paper, we show that $D^w = \bar{D}[X]_{N_v} \cap K$ and $D^w[X]_{N_v} = \bar{D}[X]_{N_v}$. Using this result, we give several w-integral closure analogs of the integral closure. We also study the w-integral closure of UMT-domains and strong Mori domains.

Keywords: w-integral closure D^w; $D[X]_{N_v(D)}$; UMT-domain; SM-domain; t-linked overring

Introduction

Let D be an integral domain with quotient field K. An overring of D means a ring between D and K. It is well known that an element $u \in K$ is integral over D if and only if $uI \subseteq I$ for some nonzero finitely generated ideal I of D (cf. [22, Theorem 12]) and that u is almost integral over D if and only if there is a nonzero ideal I of D such that $uI \subseteq I$

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(cf. [16, Lemma 3.1]). Recall that an element $u \in K$ is *pseudo-integral* over D if $uI_v \subseteq I_v$ for some nonzero finitely generated ideal I of D (see [7]). Recently, Wang introduced another type of integrality. As in [29], we say that an element $u \in K$ is *w-integral* over D if $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of D. Let $D^w = \{x \in K \mid x$ is w-integral over $D\}$. It is known that D^w is an integrally closed overring of D (see [29, §3] or Theorem 1.3); D^w is called the *w-integral closure* of D. If $D = D^w$, we say that D is *w-integrally closed*. It is clear that "u integral $\Rightarrow$ u w-integral $\Rightarrow$ u pseudo-integral $\Rightarrow$ u almost integral," and therefore $D \subseteq \bar{D} \subseteq D^w \subseteq \tilde{D} \subseteq D^*$, where $\bar{D}$ (respectively, D^w, $\tilde{D}$, D^*) is the integral closure (respectively, w-integral closure, pseudo-integral closure, complete integral closure) of D. Moreover, D completely integrally closed $\Rightarrow$ D pseudo-integrally closed $\Rightarrow$ D w-integrally closed $\Rightarrow$ D integrally closed; if D is a Noetherian domain, then $\bar{D} = D^w = \tilde{D} = D^*$; if D is an SM-domain, then $D^w = \tilde{D} = D^*$; and if D is a Mori domain, then $\tilde{D} = D^*$. (Definitions follow.)

To facilitate the reading of the introduction and of the paper, we first review basic facts on $*$-operations. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. A mapping $*: \mathcal{F}(D) \to \mathcal{F}(D)$ is called a $*$-operation on D if $*$ satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathcal{F}(D)$:

(1) $D^* = D$ and $(aI)^* = aI^*$,
(2) $I \subseteq I^*$, and if $I \subseteq J$, then $I^* \subseteq J^*$, and
(3) $(I^*)^* = I^*$.

Given two fractional ideals $I, J \in \mathcal{F}(D)$, we have $(IJ)^* = (I^*J)^* = (I^*J^*)^*$. An $I \in \mathcal{F}(D)$ is called a *$*$-ideal* if $I^* = I$. A $*$-ideal $I \in \mathcal{F}(D)$ is of *finite type* if $I = (a_1, \ldots, a_n)^*$ for some $(0) \neq (a_1, \ldots, a_n) \subseteq I$. An $I \in \mathcal{F}(D)$ is said to be $*$-invertible if $(II^{-1})^* = D$. Let $*$-$\mathrm{Max}(D)$ denote the set of $*$-ideals of D maximal among proper integral $*$-ideals of D. A $*$-operation is said to be of *finite character* if $I^* = \bigcup J^*$, where J ranges over all nonzero finitely generated subideals of I. It is easily verified that if $*$ is a finite character $*$-operation on D, then $*$-$\mathrm{Max}(D) \neq \emptyset$ and every nonzero integral ideal is contained in a maximal $*$-ideal when D is not a field. The most famous $*$-operations are the so-called v-operation and t-operation. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = (D : I) = \{x \in K \mid xI \subseteq D\}$, whereas the t-operation is $I_t = \bigcup\{J_v \mid (0) \neq J \subseteq I$ is a finitely generated deal$\}$. The w-operation is a mapping $I \mapsto I_w = \{x \in K \mid Jx \subseteq I$ for some $J \in \mathrm{GV}(D)\}$, where $\mathrm{GV}(D)$ is the set of finitely generated ideals J of D with $J^{-1} = D$. Clearly $I_w \subseteq I_t \subseteq I_v$ (and hence v-ideals are t-ideals and t-ideals are w-ideals). It is known that w-$\mathrm{Max}(D) = t$-$\mathrm{Max}(D)$ and $I_w = \bigcap_{P \in t\text{-}\mathrm{Max}(D)} ID_P = ID[X]_{N_v} \cap K$ ([1, Corollaries 2.13 and 2.17] and [12, Lemma 2.1(2)]). Recall that D is a *Mori domain* (respectively, *strong Mori domain* (SM-domain)) if D satisfies the ascending chain condition on integral v-ideals (respectively, w-ideals). Also, recall that D is a *Prüfer v-multiplication domain* (PVMD) if every finitely generated ideal $I \in \mathcal{F}(D)$ is t-invertible.

Let D be an integral domain. Throughout this paper, $\mathrm{qf}(D)$ denotes the quotient field of D; $\bar{D}$ (respectively, D^w, $\tilde{D}$, D^*) is the integral closure (respectively, w-integral closure, pseudo-integral closure, complete integral closure) of D in $\mathrm{qf}(D)$; X is an indeterminate over D; $D[X]$ is the polynomial ring over D; the *content* of a polynomial $f \in \mathrm{qf}(D)[X]$, denoted by A_f, is the fractional ideal of D generated by the coefficients of f; and $N_v(D) = \{f \in D[X] \mid (A_f)_v = D\}$.

In Section 1, we show that D^w is t-linked over D; $D^w = \bar{D}[X]_{N_v(D)} \cap \mathrm{qf}(D)$; and $\bar{D}[X]_{N_v(D)} = D^w[X]_{N_v(D)}$. Using these results, we give several w-integral closure analogs of the integral closure. We also construct an integral domain R such that $R \subsetneq \bar{R} \subsetneq R^w \subsetneq \tilde{R} \subsetneq R^*$. It is known that D is a UMT-domain if and only if $\bar{D}[X]_{N_v(D)}$ is a Prüfer domain [15, Theorem 2.5], if and only if D^w is a w_D-multiplication domain [29, Theorem 4.2]. (See Lemma 2.3 for the definition of the w_D-operation.) We prove in Section 2 that D is a UMT-domain if and only if D^w is a PVMD and $t\text{-Max}(D^w) = \{Q \in \mathrm{Spec}(D^w) \mid Q \cap D \in t\text{-Max}(D)\}$, if and only if D^w is a PVMD and $D^w[X]_{N_v(D)} = D^w[X]_{N_v(D^w)}$, if and only if D^w is a PVMD and the pair D, D^w satisfies the property that for a prime ideal Q of D^w, $(Q \cap D)_t \subsetneq D$ implies $Q_t \subsetneq D^w$. Recall that if D is a Noetherian domain, then $\bar{D} = D^*$ and D^* is a Krull domain [26, Theorem 33.10]. In Section 3, we generalize this fact to SM-domains, i.e., we show that if D is an SM-domain, then $D^w = D^*$ and D^* is a Krull domain. We also prove that if D is a weakly (respectively, an almost weakly) factorial SM-domain, then D^* is a factorial (respectively, an almost factorial) domain and that if D is a Noetherian domain with $\dim(D) = 1$, then D is a weakly (respectively, an almost weakly) factorial domain if and only if each overring of D is a weakly (respectively, an almost weakly) factorial domain.

1. w-integrality

Let $D \subseteq R$ be an extension of integral domains. Then R is said to be *t-linked over D* if, for each finitely generated ideal I of D, $I^{-1} = D$ implies $(IR)^{-1} = R$. Recall that if R is an overring of D, then R is t-linked over D if and only if, for each prime t-ideal Q of R, $(Q \cap D)_t \subsetneq D$ [14, Proposition 2.1], if and only if $R = R[X]_{N_v(D)} \cap \mathrm{qf}(R)$ [12, Lemma 3.2].

We begin this section with some characterizations of "t-linkedness," which are essential in the subsequent arguments.

Proposition 1.1. *Let $D \subseteq R$ be an extension of integral domains.*

(1) *The following statements are equivalent.*
  - (a) *R is t-linked over D.*
  - (b) *For each prime t-ideal Q of R with $Q \cap D \neq (0)$, $(Q \cap D)_t \subsetneq D$.*
  - (c) *$R = R[X]_{N_v(D)} \cap \mathrm{qf}(R)$, where X is an indeterminate over R.*

(2) *If R is an overring of D, then R is t-linked over D if and only if R is defined by a family of overrings $\{R_\alpha\}$ such that each R_α contains D_P for some maximal t-ideal P of D.*

Proof. (1)(a) $\Leftrightarrow$ (b). See [2, Proposition 2.1]. (a) $\Leftrightarrow$ (c). See the proof of [12, Lemma 3.2].

(2)($\Rightarrow$) Recall that R is t-linked over D if and only if $R = \bigcap R_{D \setminus P}$, where P ranges over the prime t-ideals of D [14, Proposition 2.13]. Thus $\{R_{D \setminus P} \mid P$ is a prime t-ideal of $D\}$ is a desired family of overrings of R.

($\Leftarrow$) Suppose that R is defined by a family of overrings $\{R_\alpha\}$ such that each R_α contains D_P for some $P \in t\text{-Max}(D)$. Let A be a nonzero finitely generated ideal of D with $A^{-1} = D$. Then $A \nsubseteq P$, and hence $AD_P = D_P$ for all $P \in t\text{-Max}(D)$. So $D_P = AD_P \subseteq AR_\alpha$ for some $P \in t\text{-Max}(D)$. But this forces $1 \in AR_\alpha$ ensuring $AR_\alpha = R_\alpha$. Now, let $*$ be the star operation on R induced by $\{R_\alpha\}$ [17, Theorem 32.5]. Then for A with the above description, we have $R = (AR)^* \subseteq (AR)_v \subseteq R$. Hence $(AR)_v = R$, and thus $(AR)^{-1} = R$. $\square$

Lemma 1.2. *The w-integral closure D^w of D is t-linked over D.*

Proof. Let $K = \text{qf}(D)$, $N_v = N_v(D)$, and R an overring of D. Note that R is t-linked over D if and only if $R = R[X]_{N_v} \cap K$ (Proposition 1.1(1)); so it suffices to show that $D^w = D^w[X]_{N_v} \cap K$. Clearly $D^w \subseteq D^w[X]_{N_v} \cap K$. For the reverse containment, let $u = f/g \in D^w[X]_{N_v} \cap K$, where $g \in N_v$ and $f \in D^w[X]$. Let $f = a_0 + a_1X + \cdots + a_nX^n \in D^w[X]$. Since each $a_i \in D^w$, there is a nonzero finitely generated ideal J_i of D such that $a_i(J_i)_w \subseteq (J_i)_w$. Let $I = J_1 \cdots J_n$. Then I is a nonzero finitely generated ideal such that $a_iI_w = (a_iJ_iJ_1 \cdots J_{i-1}J_{i+1} \cdots J_n)_w \subseteq (J_iJ_1 \cdots J_{i-1}J_{i+1} \cdots J_n)_w = I_w$. So $A_fI_w \subseteq I_w$, and hence $uI_w = u((A_g)_wI)_w = u(A_gI)_w = (uA_gI)_w = (A_fI)_w = (A_fI_w)_w \subseteq (I_w)_w = I_w$ (note that $(A_g)_w = D$ since $t\text{-Max}(D) = w\text{-Max}(D)$ [1, Corollary 2.17]). Therefore, $u \in D^w$. $\square$

Let D be a Noetherian domain. Then $\bar{D} = D^w = D^*$, and thus Lemma 1.2 gives another proof of the well-known fact that $\bar{D}$ is t-linked over D [14, Corollary 2.3]. We next give the main result of this section, which is very useful for the study of w-integrality. This result also shows that D^w is the smallest integrally closed t-linked overring of D [14, Proposition 2.13(b)] and that $\bar{D}$ is t-linked over D if and only if $\bar{D} = D^w$ (Proposition 1.1(1)).

Theorem 1.3. *Let K be the quotient field of D and $N_v = \{f \in D[X] \mid (A_f)_v = D\}$.*

(1) $D^w = \bar{D}[X]_{N_v} \cap K = \bigcap_{P \in t\text{-Max}(D)} \bar{D}_{D\setminus P}$.
(2) $D^w[X]_{N_v} = \bar{D}[X]_{N_v}$.

Proof. Note that $\bar{D}[X]_{N_v}$ is the integral closure of $D[X]_{N_v}$ since $\bar{D}[X]$ is the integral closure of $D[X]$.

(1) (Proof of $D^w = \bar{D}[X]_{N_v} \cap K$.) Let $u \in D^w$. Then $uI_w \subseteq I_w$ for some nonzero finitely generated ideal I of $D \Rightarrow uID[X]_{N_v} = uI_wD[X]_{N_v} \subseteq I_wD[X]_{N_v} = ID[X]_{N_v}$ [12, Lemma 2.1(2)] $\Rightarrow u$ is integral over $D[X]_{N_v}$ [22, Theorem 12] $\Rightarrow u \in \bar{D}[X]_{N_v} \cap K$. So $D^w \subseteq \bar{D}[X]_{N_v} \cap K$. Note that $\bar{D} \subseteq D^w$ and D^w is t-linked over D (Lemma 1.2). Hence $D^w \subseteq \bar{D}[X]_{N_v} \cap K \subseteq D^w[X]_{N_v} \cap K = D^w$ (Proposition 1.1(1)), and thus $D^w = \bar{D}[X]_{N_v} \cap K$.

(Proof of $\bar{D}[X]_{N_v} \cap K = \bigcap_{P \in t\text{-Max}(D)} \bar{D}_{D \setminus P}$.) Let $R = \bigcap_{P \in t\text{-Max}(D)} \bar{D}_{D \setminus P}$. Then $\bar{D} \subseteq R$ and R is t-linked over D (Proposition 1.1(2)); so $\bar{D}[X]_{N_v} \cap K \subseteq R[X]_{N_v} \cap K = R$ (Proposition 1.1(1)). For the reverse containment, note that

$$\bar{D}[X]_{N_v} = \bigcap_{P \in t\text{-Max}(D)} \left(\bar{D}[X]_{N_v}\right)_{(D[X]_{N_v} \setminus P[X]_{N_v})}$$

since $\bar{D}[X]_{N_v}$ is integral over $D[X]_{N_v}$ and $\mathrm{Max}(D[X]_{N_v}) = \{P[X]_{N_v} \mid P \in t\text{-Max}(D)\}$ [21, Proposition 2.1(2)]. Also, note that

$$\left(\bar{D}[X]_{N_v}\right)_{(D[X]_{N_v} \setminus P[X]_{N_v})} = \bar{D}[X]_{D[X] \setminus P[X]} \supseteq \bar{D}_{D \setminus P} \quad \text{for all } P \in t\text{-Max}(D).$$

Thus $\bar{D}[X]_{N_v} \cap K \supseteq \bigcap_{P \in t\text{-Max}(D)} \bar{D}_{D \setminus P} = R$.

(2) Let $f/g \in D^w[X]_{N_v}$, where $f \in D^w[X]$ and $g \in N_v$. Note that g and powers of X are units of $D[X]_{N_v}$ and that the coefficients of f are integral over $D[X]_{N_v}$ (see the proof of (1)). Hence f/g is integral over $D[X]_{N_v}$ [22, Theorem 13], and thus $f/g \in \bar{D}[X]_{N_v}$. The reverse containment is clear because $\bar{D} \subseteq D^w$. □

As in [22, p. 28], INC, GU, and LO denote incomparable, going up, and lying over, respectively. It is well known that if $R \subseteq T$ are rings with T integral over R, then the pair R, T satisfies INC, GU, and LO [22, Theorem 44].

Corollary 1.4.

(1) $(D^w)^w = D^w$ *and* $(\bar{D})^w = \bar{D}$.
(2) *If R is a t-linked overring of D, then $D^w \subseteq R^w$.*
(3) *The pair D, D^w satisfies INC, GU, and LO for prime w-ideals of D.*
(4) *D^w is the intersection of t-linked valuation overrings of D.*

Proof. Let $K = \mathrm{qf}(D)$ and $N_v = N_v(D)$. Note that $D^w = \bar{D}[X]_{N_v} \cap K$; $D^w[X]_{N_v} = \bar{D}[X]_{N_v}$; and $\bar{D}[X]_{N_v}$ is the integral closure of $D[X]_{N_v}$ (Theorem 1.3).

(1) Since D^w and $\bar{D}$ are integrally closed,

$$\left(D^w\right)^w = \overline{D^w}[X]_{N_v(D^w)} \cap K = D^w[X]_{N_v(D^w)} \cap K = D^w \quad \text{and}$$
$$\left(\bar{D}\right)^w = \bar{\bar{D}}[X]_{N_v(\bar{D})} \cap K = \bar{D}[X]_{N_v(\bar{D})} \cap K = \bar{D}.$$

(2) Note that $\bar{D} \subseteq \bar{R}$ and $N_v \subseteq N_v(R)$; so $\bar{D}[X]_{N_v} \subseteq \bar{R}[X]_{N_v(R)}$. Thus $D^w = \bar{D}[X]_{N_v} \cap K \subseteq \bar{R}[X]_{N_v(R)} \cap K = R^w$.

(3) (LO) Let P be a prime ideal of D such that $P_w \subsetneq D$. Then as $P_t \subsetneq D$ (cf. [1, Corollary 2.17]), we have $P[X] \cap N_v = \emptyset$; hence $P[X]_{N_v}$ is a proper prime ideal of $D[X]_{N_v}$. So by [22, Theorem 44], there is a prime ideal A of $D^w[X]_{N_v}$ lying over $P[X]_{N_v}$ since $D^w[X]_{N_v}$ is integral over $D[X]_{N_v}$. Thus $A \cap D^w$ is a prime ideal of D^w such that $(A \cap D^w) \cap D = P$.

The properties of INC and GU also follow directly from [22, Theorem 44], Lemma 1.2, and the fact that $D^w[X]_{N_v}$ is integral over $D[X]_{N_v}$.

(4) Note that $D^w[X]_{N_v}$ is the integral closure of $D[X]_{N_v}$; so $D^w[X]_{N_v}$ is the intersection of valuation overrings of $D[X]_{N_v}$ [17, Theorem 19.8]. Let $\{W_\alpha\}$ be the set of valuation overrings of $D[X]_{N_v}$, and let $V_\alpha = W_\alpha \cap K$. Then each V_α is a valuation overring of D [17, Theorem 19.16] and $D^w = D^w[X]_{N_v} \cap K = (\bigcap_\alpha W_\alpha) \cap K = \bigcap_\alpha (W_\alpha \cap K) = \bigcap_\alpha V_\alpha$. Moreover, note that $D^w[X]_{N_v} \subseteq V_\alpha[X]_{N_v} \subseteq W_\alpha$; so $V_\alpha \subseteq V_\alpha[X]_{N_v} \cap K \subseteq W_\alpha \cap K = V_\alpha$, and hence $V_\alpha[X]_{N_v} \cap K = V_\alpha$. Thus V_α is t-linked over D by Proposition 1.1(1). □

Remark 1.5. (1) Corollary 1.4(1) shows that D is integrally closed if and only if D is w-integrally closed.

(2) Let D be an integral domain such that $\bar{D}$ is not t-linked over D (see, for example, [13, Example 4.1]). Then $\bar{D} \subsetneq D^w$ (Lemma 1.2) and $(\bar{D})^w = \bar{D}$ (Corollary 1.4(1)); hence $(\bar{D})^w \subsetneq D^w$. Thus for Corollary 1.4(2), we need the assumption that R is t-linked over D.

(3) Corollary 1.4(3) is a special case of [29, Theorem 3.3].

(4) Initially Wang considered w-integrality in [28], but we follow the definition given in [29].

Lemma 1.6. *Let R be an overring of D and $\{X_\alpha\}$ a nonempty set of indeterminates over D. Then R is t-linked over D if and only if $R[\{X_\alpha\}]$ is t-linked over $D[\{X_\alpha\}]$.*

Proof. This follows directly from the following fact: Let S be an integral domain, and let $\{X_\beta\}$ be a nonempty set of indeterminates over S. Let Q be a maximal t-ideal of $S[\{X_\beta\}]$. If $Q \cap S = (0)$, then $\operatorname{ht} Q = 1$ (see, for example, [12, proof of Theorem 1.4]). If $Q \cap S \neq (0)$, then $Q = (Q \cap S)[\{X_\beta\}]$ and $Q \cap S$ is a maximal t-ideal of S [15, Proposition 2.2]. □

Let D be an integral domain such that $\bar{D}$ is not t-linked over D. Then since $\bar{D}[\{X_\alpha\}]$ is the integral closure of $D[\{X_\alpha\}]$, by Lemma 1.6 the integral closure of $D[\{X_\alpha\}]$ is not t-linked over $D[\{X_\alpha\}]$. The following result is another nice property of w-integrality.

Proposition 1.7. *Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D. Then $(D[\{X_\alpha\}])^w = D^w[\{X_\alpha\}]$.*

Proof. ($\subseteq$) Let t be an indeterminate over $D[\{X_\alpha\}]$, $S = N_v(D[\{X_\alpha\}])$, and $K(\{X_\alpha\}) = \operatorname{qf}(D[\{X_\alpha\}])$. Then

$$\begin{aligned}\big(D\big[\{X_\alpha\}\big]\big)^w &= \big(\overline{D\big[\{X_\alpha\}\big]}\big)[t]_S \cap K\big(\{X_\alpha\}\big) = \big(\bar{D}\big[\{X_\alpha\}\big]\big)[t]_S \cap K\big(\{X_\alpha\}\big) \\ &\subseteq \big(D^w\big[\{X_\alpha\}\big]\big)[t]_S \cap K\big(\{X_\alpha\}\big) = D^w\big[\{X_\alpha\}\big]\end{aligned}$$

by Proposition 1.1(1) and Lemma 1.6.

($\supseteq$) Let $u \in D^w$. Then there is a nonzero finitely generated ideal J of D such that $uJ_w \subseteq J_w$. Now, since $J_w[\{X_\alpha\}] = (J[\{X_\alpha\}])_w$ (cf. [18, Proposition 4.3] for one indeterminate), we have $u \in (D[\{X_\alpha\}])^w$. So $D^w \subseteq (D[\{X_\alpha\}])^w$, and thus $D^w[\{X_\alpha\}] \subseteq (D[\{X_\alpha\}])^w$. □

We end this section by using the $D+M$ construction to construct an integral domain R such that $R \subsetneq \bar{R} \subsetneq R^w \subsetneq \tilde{R} \subsetneq R^*$.

Example 1.8. Let D be an integral domain with quotient field K such that $\bar{D}$ is not t-linked over D (see [13, Example 4.1] for such an integral domain), and let $K \subsetneq F$ be an algebraic field extension. Let V be a two-dimensional valuation domain of the form $F(t)+M$, where t is an indeterminate over F. Finally, let $R = D+M$ and D_1 the integral closure of D in F. Then

(1) $\bar{R} = D_1 + M$ and $\tilde{R} = V$.
(2) D_1 is not t-linked over D; so $D_1 + M$ is not t-linked over R.
(3) $\bar{R} \subsetneq R^w \subseteq F + M \subsetneq V$.
(4) $V \subsetneq R^*$.

Therefore, $R \subsetneq \bar{R} \subsetneq R^w \subsetneq \tilde{R} \subsetneq R^*$.

Proof. (1) See [9, Theorem 2.1(b)] for $\bar{R} = D_1 + M$ (note that F is algebraically closed in $F(t)$) and [7, Proposition 1.8(ii)] for $\tilde{R} = V$.

(2) Suppose that D_1 is t-linked over D. Then $D_1[X]_{N_v(D)} \cap F = D_1$ (Proposition 1.1(1)), and hence $D^w = \bar{D}[X]_{N_v(D)} \cap K \subseteq D_1[X]_{N_v(D)} \cap K = (D_1[X]_{N_v(D)} \cap F) \cap K = D_1 \cap K = \bar{D}$ (Theorem 1.3). So $D^w = \bar{D}$, which is contrary to the fact that $\bar{D} \subsetneq D^w$ (Lemma 1.2). Therefore, D_1 is not t-linked over D.

Recall that if I is a nonzero ideal of D (respectively, D_1), then $(I+M)_t = I_t + M$ (cf. [8, Proposition 2.4]); so P is a maximal t-ideal of D (respectively, D_1) if and only if $P+M$ is a maximal t-ideal of $D+M$ (respectively, D_1+M). Since D_1 is not t-linked over D, there is a prime t-ideal Q of D_1 such that $(Q \cap D)_t = D$ (Proposition 1.1(1)). Hence $Q+M$ is a prime t-ideal of D_1+M such that $((Q+M) \cap R)_t = ((Q \cap D)+M)_t = (Q \cap D)_t + M = D+M = R$. Thus D_1+M is not t-linked over R.

(3) Since $\bar{R} = D_1+M$ by (1) and $\bar{R}$ is not t-linked over R by (2), we have $\bar{R} \subsetneq R^w$ by Lemma 1.2. Note that $F+M$ is quasi-local with maximal ideal M and M is a t-ideal of R [8, Proposition 2.1(3)]; so $F+M$ is t-linked over R. Therefore, $R^w = \bar{R}[X]_{N_v(R)} \cap \mathrm{qf}(R) \subseteq (F+M)[X]_{N_v(R)} \cap \mathrm{qf}(R) = F+M \subsetneq V$ by Theorem 1.3 and Proposition 1.1(1).

(4) Since M is an ideal of V, $VM \subseteq M$, and hence $V \subseteq R^*$. Let $a, b \in M$ such that $(0) \neq \sqrt{bV} \subsetneq \sqrt{aV} = M$. Then for each positive integer n, there is an $x_n \in V$ such that $b = a^n x_n$. Since $a^n \notin \sqrt{bV}$ and $\sqrt{bV}$ is a prime ideal of V, we have that $x_n \in \sqrt{bV} \subseteq M \subseteq R$. Hence $1/a$ is almost integral over R, and $1/a \in R^* \setminus V$. Therefore, $V \subsetneq R^*$ □

2. The w-integral closure of UMT-domains

Let $D \subseteq R$ be an extension of integral domains. Then D is said to be *t-linked under R* if whenever $0 \neq a_1, \ldots, a_n \in D$ with $((a_1, \ldots, a_n)R)_v = R$, then $((a_1, \ldots, a_n)D)_v = D$. The concept of "$t$-linked under" was introduced by Anderson and Zafrullah [5] when R is an overring of D. It is easy to see that D is t-linked under R if and only if $N_v(R) \cap D[X] \subseteq N_v(D)$, if and only if $(PR)_t \subsetneq R$ for each prime t-ideal P of D. Recall that R is t-linked over D if and only if $(Q \cap D)_t \subsetneq D$ for each prime t-ideal Q of R with $Q \cap D \neq (0)$

(Proposition 1.1(1)), and note that "t-linked under" sounds like the converse of the notion of "t-linked over." So it is natural to ask if D t-linked under R is equivalent to the following condition:

(#) for each prime ideal Q of R with $Q \cap D \neq (0)$, $(Q \cap D)_t \subsetneq D$ implies $Q_t \subsetneq R$.

However, there is no relationship between "t-linked under" and the property (#). For example, let $P = 2\mathbb{Z}$, and let $R = \mathbb{Z}_P$. Then the pair $\mathbb{Z}$, R satisfies (#), but $\mathbb{Z}$ is not t-linked under R (note that $(3\mathbb{Z}R)_t = R$ and $(3\mathbb{Z})_t \subsetneq \mathbb{Z}$). Let D be a two-dimensional local Noetherian domain with maximal ideal P such that $P_t = P$ and $\bar{D}$ has a height-one prime ideal lying over P. Then D is t-linked under $\bar{D}$, but the pair D, $\bar{D}$ does not satisfy (#). However, as we shall see in the sequel, the property (#) does play an important role. We next give an explicit example (for more details of this example, see [6, Remark 2.7(b)] and [20, Example 28]).

Example 2.1. Let X, Y be indeterminates over $\mathbb{C}$ and $S = \mathbb{C}[X, Y]$. Let $T_1 \subseteq \mathbb{C}(X, Y)$ be a DVR with maximal ideal P such that $T_1 = \mathbb{C} + P$ with $(X, Y)S \subseteq P$, and let $T_2 = \mathbb{C}[X, Y]_{(X-1,Y)}$. Then $T = T_1 \cap T_2$ is a two-dimensional Noetherian factorial domain with exactly two maximal ideals, $M = P \cap T$ and $N = (X - 1, Y)_{(X-1,Y)} \cap T$, where $\operatorname{ht} M = 1$ and $\operatorname{ht} N = 2$. Note that $T = \mathbb{C} + M$, and let $R = \mathbb{C} + (M \cap N)$. Then R is a two-dimensional local Noetherian domain with maximal ideal $M \cap N$ and $\bar{R} = T$.

Let $D = \mathbb{R} + (M \cap N)$. Then D is a Noetherian domain [11, Theorem 4] (since $[\mathbb{C} : \mathbb{R}] = 2 < \infty$), $\operatorname{ht}(M \cap N) = 2$, and $(M \cap N)_t \subsetneq D$. It is clear that R is integral over D and $\operatorname{qf}(D) = \operatorname{qf}(R)$; so D is local (cf. [22, Theorem 44]) and $T = \bar{D}$ [17, Corollary 9.5] because $T = \bar{R}$. Since $\operatorname{ht} M = 1$, $M_t = M$ in T, and thus $((M \cap N)T)_t \subseteq M_t = M \subsetneq T$. Therefore, D is t-linked under T. But the pair D, T does not satisfy (#) since $N \cap D = M \cap N$ is a prime t-ideal of D, but $N_t = T$ (note that T is a UFD and $\operatorname{ht} N = 2$).

An extension $D \subseteq R$ of integral domains is called a *root extension* if, for each $x \in R$, there is a positive integer $n = n(x)$ such that $x^n \in D$. It is well known that D is an almost GCD-domain if and only if $\bar{D}$ is an almost GCD-domain, $D \subseteq \bar{D}$ is a root extension, and D is t-linked under $\bar{D}$ [5, Theorem 5.9]. (Recall that D is an *almost GCD-domain* if for $0 \neq x, y \in D$, there exists a positive integer $n = n(x, y)$ such that $x^n D \cap y^n D$ is principal.)

Proposition 2.2. *Let R be an overring of D.*

(1) *If $R \subseteq D^w$ and the pair D, R satisfies (#), then D is t-linked under R.*
(2) *If R is a root extension of D, then D is t-linked under R if and only if the pair D, R satisfies (#).*

Proof. (1) Let P be a prime t-ideal of D. Then P is a prime w-ideal of D, and hence there is a prime ideal Q of R such that $Q \cap D = P$ (cf. Corollary 1.4(3)); so $Q_t \subsetneq R$ by assumption. Hence $(PR)_t \subseteq Q_t \subsetneq R$, and thus D is t-linked under R.

(2) Assume that D is t-linked under R, and let Q be a nonzero prime ideal of R such that $(Q \cap D)_t \subsetneq D$. If $Q_t = R$, then there are some $0 \neq x_1, \ldots, x_k \in Q$ such that

$((x_1, \ldots, x_k)R)_v = R$. Also, since $D \subseteq R$ is a root extension, there is a positive integer n such that $x_i^n \in D$. Clearly $((x_1^n, \ldots, x_k^n)R)_v = R$, and hence $((x_1^n, \ldots, x_k^n)D)_v = D$ by assumption; so $(Q \cap D)_t = D$, a contradiction. Thus $Q_t \subsetneq R$. The converse is an immediate consequence of (1) because $R \subseteq \bar{D} \subseteq D^w$. □

Lemma 2.3. *Let R be a t-linked overring of D and K the quotient field of D. Then the mapping $I \mapsto I_{w_D} = IR[X]_{N_v(D)} \cap K$ is a finite character $*$-operation on R.*

Proof. Let $0 \neq a \in K$ and let I, J be nonzero fractional ideals of R. First, note that $R = R[X]_{N_v(D)} \cap K$ (Proposition 1.1(1)) and

$$(aI)_{w_D} = aIR[X]_{N_v(D)} \cap K = a\big(IR[X]_{N_v(D)} \cap K\big) = aI_{w_D};$$

so I_{w_D} is a nonzero fractional ideal of R and $(aI)_{w_D} = aI_{w_D}$. It is clear that $I \subseteq I_{w_D}$; if $I \subseteq J$, then $I_{w_D} \subseteq J_{w_D}$; and $(I_{w_D})_{w_D} = I_{w_D}$. Hence w_D is a $*$-operation on R. Next, to show that w_D is of finite character, let $\{J_\alpha\}$ be the set of nonzero finitely generated subideals of I. Then $I = \bigcup_\alpha J_\alpha$ and

$$\begin{aligned} I_{w_D} &= IR[X]_{N_v(D)} \cap K = \left(\bigcup_\alpha J_\alpha\right) R[X]_{N_v(D)} \cap K \\ &= \left(\bigcup_\alpha \big(J_\alpha R[X]_{N_v(D)}\big)\right) \cap K = \bigcup_\alpha \big(J_\alpha R[X]_{N_v(D)} \cap K\big) = \bigcup_\alpha (J_\alpha)_{w_D}. \end{aligned}$$

Thus the w_D-operation is a finite character $*$-operation on R. □

Let R be a t-linked overring of D, and let I be a nonzero fractional ideal of R. Then $I_{w_D} = IR[X]_{N_v(D)} \cap \mathrm{qf}(R) \subseteq IR[X]_{N_v(R)} \cap \mathrm{qf}(R) = I_w$ [12, Lemma 2.1(2)] since $N_v(D) \subseteq N_v(R)$. Moreover, if $R = D$, then $I_{w_D} = I_w$.

Lemma 2.4. *The following statements are equivalent for an integral domain D.*

(1) *The pair D, D^w satisfies* (#).
(2) $t\text{-Max}(D^w) = \{Q \in \mathrm{Spec}(D^w) \mid Q \cap D \in t\text{-Max}(D)\}$.
(3) $D^w[X]_{N_v(D)} = D^w[X]_{N_v(D^w)}$.
(4) $t\text{-Max}(D^w) = w_D\text{-Max}(D^w)$.

Proof. (1) ⇒ (2). First, note that for each prime t-ideal Q of D^w, $(Q \cap D)_t \subsetneq D$ (Proposition 1.1(1)) since D^w is t-linked over D (Lemma 1.2). Thus if $Q \cap D \in t\text{-Max}(D)$, then $Q \in t\text{-Max}(D^w)$ by Corollary 1.4(3). For the reverse containment, assume that Q is a maximal t-ideal of D^w, and let $P \in t\text{-Max}(D)$ containing $Q \cap D$. By Corollary 1.4(3), there is a prime ideal Q' of D^w such that $Q \subseteq Q'$ and $Q' \cap D = P$. So $Q = Q'$ since $Q'_t \subsetneq D^w$ by (1) and Q is a maximal t-ideal. Thus $Q \cap D = P$.

(2) ⇒ (3). For easy reference, we first recall that for any integral domain R, $\mathrm{Max}(R[X]_{N_v(R)}) = \{P[X]_{N_v(R)} \mid P \in t\text{-Max}(R)\}$ [21, Proposition 2.1]. Let A be a maximal ideal of $D^w[X]_{N_v(D)}$, and let $Q = A \cap D^w$. Then there is a maximal t-ideal P of

D such that $A \cap (D[X]_{N_v(D)}) = P[X]_{N_v(D)}$ [22, Theorem 44] since $D^w[X]_{N_v(D)}$ is integral over $D[X]_{N_v(D)}$ (Theorem 1.3(2)). It is clear that $A \cap D = Q \cap D = P$. Hence $Q[X]_{N_v(D)} \cap D[X]_{N_v(D)} = P[X]_{N_v(D)}$, and thus $A = Q[X]_{N_v(D)}$ [22, Theorem 44]. This implies that $\mathrm{Max}(D^w[X]_{N_v(D)}) = \{Q[X]_{N_v(D)} \mid Q \in t\text{-}\mathrm{Max}(D^w)\}$ by (2). Therefore,

$$\begin{aligned} D^w[X]_{N_v(D)} &= \bigcap_{Q \in t\text{-}\mathrm{Max}(D^w)} \big(D^w[X]_{N_v(D)}\big)_{(Q[X]_{N_v(D)})} \\ &= \bigcap_{Q \in t\text{-}\mathrm{Max}(D^w)} \big(D^w[X]\big)_{Q[X]} \\ &= \bigcap_{Q \in t\text{-}\mathrm{Max}(D^w)} \big(D^w[X]_{N_v(D^w)}\big)_{(Q[X]_{N_v(D^w)})} = D^w[X]_{N_v(D^w)}. \end{aligned}$$

(3) $\Rightarrow$ (1). Let $Q \in \mathrm{Spec}(D^w)$ such that $(Q \cap D)_t \subsetneq D$. Then $Q[X] \cap N_v(D) = \emptyset$, and hence $Q[X] \cap N_v(D^w) = \emptyset$ by (3). Thus $Q_t \subsetneq D^w$.

(1) $\Rightarrow$ (4). Since D^w is t-linked over D (Lemma 1.2), w_D is a finite character $*$-operation on D^w (Lemma 2.3), and hence every maximal t-ideal of D^w is a w_D-ideal. So it suffices to show that if Q is a maximal w_D-ideal, then $Q_t \subsetneq D^w$. Let $Q \in w_D\text{-}\mathrm{Max}(D^w)$. Then $Q[X]_{N_v(D)} \subsetneq D^w[X]_{N_v(D)}$ since $Q = Q[X]_{N_v(D)} \cap \mathrm{qf}(D)$; so $(Q \cap D)[X]_{N_v(D)} \subsetneq D[X]_{N_v(D)}$. Hence $(Q \cap D)_t \subsetneq D$ (cf. [21, Proposition 2.1]), and thus $Q_t \subsetneq D^w$ by (1).

(4) $\Rightarrow$ (1). Let Q be a prime ideal of D^w such that $(Q \cap D)_t \subsetneq D$. Then $Q[X] \cap N_v(D) = \emptyset$, and hence $Q_{w_D} = Q[X]_{N_v(D)} \cap \mathrm{qf}(D) = Q \subsetneq D^w$. Thus $Q_t \subsetneq D^w$ by (4). $\square$

Remark 2.5. (1) Let $t\text{-}\dim(D) = 1$. If Q is a prime ideal of D^w such that $(Q \cap D)_t \subsetneq D$, then $\mathrm{ht}(Q \cap D) = 1$, and hence $\mathrm{ht}\, Q = 1$ by Corollary 1.4(3). Hence the pair D, D^w satisfies (#), and thus D is t-linked under D^w (Proposition 2.2). (Recall that the *t-dimension* of D, denoted by $t\text{-}\dim(D)$, is the length of the longest chain of prime t-ideals of D.)

(2) The proof of Lemma 2.4 shows that Lemma 2.4 holds for any t-linked overring R of D with $R \subseteq D^w$.

(3) $t\text{-}\mathrm{Max}(D^w) \supseteq \{Q \in \mathrm{Spec}(D^w) \mid Q \cap D \in t\text{-}\mathrm{Max}(D)\}$ by the proof of (1) $\Rightarrow$ (2) of Lemma 2.4.

Let $*$ be a $*$-operation on D. We say that D is a *$*$-multiplication domain* if every nonzero finitely generated ideal I of D is $*$-invertible, i.e., $(II^{-1})^* = D$. In particular, if $* = t$, then a $*$-multiplication domain is called a *Prüfer v-multiplication domain* (PVMD). It is clear that if $*$ is of finite character, then a $*$-multiplication domain is a PVMD. Recall that D is a UMT-domain if every upper to zero in $D[X]$ is a maximal t-ideal. It is well known that D is an integrally closed UMT-domain if and only if D is a PVMD [19, Proposition 3.2]. However, the integral closure of a UMT-domain need not be a PVMD (see [23, Proposition 2.7]). Furthermore, $\bar{D}$ being a PVMD does not imply that D is a UMT-domain. For example, let D be a Noetherian domain with $t\text{-}\dim(D) \geqslant 2$. Then $\bar{D}$ is a PVMD, but D is not a UMT-domain (see Remark 2.7). We next give some new characterizations of UMT-domains.

Theorem 2.6. *The following statements are equivalent for an integral domain D.*

(1) *D is a UMT-domain.*
(2) *$\bar{D}[X]_{N_v(D)}$ is a Prüfer domain.*
(3) *D^w is a w_D-multiplication domain.*
(4) *D^w is a PVMD and the pair D, D^w satisfies (#).*
(5) *D^w is a PVMD and $t\text{-Max}(D^w) = \{Q \in \operatorname{Spec}(D^w) \mid Q \cap D \in t\text{-Max}(D)\}$.*
(6) *D^w is a PVMD and $D^w[X]_{N_v(D)} = D^w[X]_{N_v(D^w)}$.*
(7) *D^w is a PVMD and $t\text{-Max}(D^w) = w_D\text{-Max}(D^w)$.*
(8) *Each t-linked overring of D is a UMT-domain.*

Proof. Let $N_v = N_v(D)$ and $K = \operatorname{qf}(D)$. Note that $D^w[X]_{N_v} = \bar{D}[X]_{N_v}$ is the integral closure of $D[X]_{N_v}$ (Theorem 1.3) and D^w is t-linked over D (Lemma 1.2).

(1) $\Leftrightarrow$ (2). This follows directly from [15, Theorem 2.5].

(1) $\Leftrightarrow$ (3). It is easy to see that $I_{w_D} = \{x \in K \mid xJ \subseteq I$ for a nonzero finitely generated ideal J of D with $J^{-1} = D\}$ for each nonzero fractional ideal I of D^w. Thus this is [29, Theorem 4.2].

(2) $\Rightarrow$ (4). Since $N_v(D) \subseteq N_v(D^w)$ (Lemma 1.2), $D^w[X]_{N_v(D^w)}$ is an overring of $D^w[X]_{N_v}$, and hence $D^w[X]_{N_v(D^w)}$ is a Prüfer domain [17, Theorem 26.1]. Thus D^w is a PVMD [21, Theorem 3.7].

Let Q be a prime ideal of D^w with $(Q \cap D)_t \subsetneq D$, and let $P \in t\text{-Max}(D)$ containing $(Q \cap D)_t$. Let Q' be a prime ideal of D^w such that $Q \subseteq Q'$ and $Q' \cap D = P$ (cf. Corollary 1.4(3)). If $Q'_t \subsetneq D^w$, then $Q_t \subsetneq D^w$. So we may assume that $Q \cap D = P$. Note that $P[X] \cap N_v = \emptyset$ and $Q[X] \cap D[X] = P[X]$; hence $Q[X] \cap N_v = \emptyset$ and $Q[X]_{N_v}$ is a proper prime ideal of $D^w[X]_{N_v}$. By (2), $(D^w[X]_{N_v})_{Q[X]_{N_v}} = D^w[X]_{Q[X]}$, and thus $D^w_Q = D^w[X]_{Q[X]} \cap K$, is a valuation domain [17, Theorem 19.16]. Thus Q_Q is a t-ideal of D^w_Q, and so $Q = Q_Q \cap D^w$ is a t-ideal of D^w [21, Lemma 3.17].

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). See Lemma 2.4.

(6) $\Rightarrow$ (2). This is an immediate consequence of [21, Theorem 3.7].

(1) $\Rightarrow$ (8). Let R be a t-linked overring of D. Then $D^w \subseteq R^w$ by Corollary 1.4(2); so $R^w[X]_{N_v}$ is an overring of $D^w[X]_{N_v}$. Thus by the equivalence of (1) and (2), D is a UMT-domain $\Rightarrow$ $D^w[X]_{N_v}$ is a Prüfer domain $\Rightarrow$ $R^w[X]_{N_v}$, and hence $R^w[X]_{N_v(R)}$, is a Prüfer domain [17, Theorem 26.1] $\Rightarrow$ R is a UMT-domain.

(8) $\Rightarrow$ (1). This is clear. □

Remark 2.7. (1) Let D be a Noetherian domain with $t\text{-dim}(D) \geqslant 2$. Then $\bar{D}$ is t-linked over D and $\bar{D}$ is a Krull domain [16, Theorem 4.3(a)] (and hence PVMD), but D is not a UMT-domain [19, Theorem 3.7]. For an explicit example, let $R = \mathbb{C}[X, Y, Z, W]/(XY - ZW) = \mathbb{C} + M$, where $M = (X, Y, Z, W)/(XY - ZW)$, and let $D = \mathbb{R} + M$. Then D is a Noetherian domain and $t\text{-dim}(D) = 3$ (see [6, Example 3.8(2)]). For another example, let K be a field, and let $D = K[Y, XY, X^2, X^3]$. Then D is a Noetherian domain with $t\text{-dim}(D) \geqslant 2$ (see [29, §6, Example]).

(2) It may be important to note that if D is a UMT-domain, then $\tilde{D}$ is a PVMD. Here is a question. Is it true that D is a UMT-domain if and only if $\tilde{D}$ is a PVMD and the pair D, $\tilde{D}$ satisfies (#).

Let Λ be a nonempty set of prime t-ideals of D. Then $\bigcap_{P\in\Lambda} D_P$ is called a *subintersection* of D. The notion of subintersection was known only for Krull domains, and then extended to arbitrary integral domains by Mott and Zafrullah [25]. Let R be an overring of D. It is well known that if D is a PVMD, then R is t-linked over D if and only if R is a subintersection of D [21, Theorem 3.8]. We next generalize this result to a UMT-domain.

Corollary 2.8. *Let D be a UMT-domain, and let R be an integrally closed overring of D. Then R is t-linked over D if and only if R is a subintersection of D^w.*

Proof. ($\Rightarrow$) Let $N_v = N_v(D)$ and $K = \mathrm{qf}(D)$. Note that D^w is the smallest integrally closed t-linked overring of D by Proposition 1.1(1) and Theorem 1.3. So if R is t-linked over D, then $D^w \subseteq R$, and hence $R[X]_{N_v}$ is an overring of $D^w[X]_{N_v}$. However, since $D^w[X]_{N_v}$ is a Prüfer domain (Theorem 2.6), $R[X]_{N_v}$ is a subintersection of $D^w[X]_{N_v}$ [17, Theorem 26.1]. Note that if Q is a prime ideal of $D^w[X]_{N_v}$, then $Q = P[X]_{N_v}$ for some prime t-ideal P of D^w by Theorem 2.6 and [21, Theorem 3.14]. Thus $R = R[X]_{N_v} \cap K = (\bigcap_{P\in\Lambda}(D^w[X]_{N_v})_{P[X]_{N_v}}) \cap K = \bigcap_{P\in\Lambda}((D^w[X])_{P[X]} \cap K) = \bigcap_{P\in\Lambda}(D^w)_P$, where Λ is a set of prime t-ideals of D^w.

($\Leftarrow$) Let S be a multiplicative subset of D^w. Then $(D^w)_S$ is t-linked over D. For if Q is a prime t-ideal of $(D^w)_S$, then $Q \cap D^w$ is a t-ideal of D^w [21, Lemma 3.17], and hence $(Q \cap D)_t = ((Q \cap D^w) \cap D)_t \subsetneq D$ (Proposition 1.1(1) and Lemma 1.2). Thus the result follows directly from [14, Proposition 2.2(b)]. $\square$

3. The complete integral closure of SM-domains

An integral domain D is called a *strong Mori domain* (SM-domain) if D satisfies the ascending chain condition on integral w-ideals; equivalently, each w-ideal of D is of finite type. It is known that D is an SM-domain if and only if D_P is Noetherian for all $P \in t\text{-}\mathrm{Max}(D)$ and each nonzero nonunit of D is contained in only a finite number of maximal t-ideals of D [30, Theorem 1.9], if and only if $D[X]_{N_v(D)}$ is a Noetherian domain [12, Theorem 2.2].

The *class group* $\mathrm{Cl}(D)$ *of* D is the group of t-invertible fractional t-ideals of D under t-multiplication modulo its subgroup of principal fractional ideals. It is well known that a Krull domain D is factorial if and only if $\mathrm{Cl}(D) = 0$ [16, Proposition 6.1]. Following [27], we call a Krull domain D an *almost factorial domain* if $\mathrm{Cl}(D)$ is torsion. Recall that D is a *weakly factorial domain* (WFD) if every nonzero nonunit of D can be written as a finite product of primary elements, while D is an *almost weakly factorial domain* (AWFD) if, for each nonunit $0 \neq x \in D$, there is a positive integer $n = n(x)$ such that x^n is a finite product of primary elements. It is well known that D is a WFD (respectively, AWFD) if and only if D is a weakly Krull domain and $\mathrm{Cl}(D) = 0$ [4, Theorem] (respectively, $\mathrm{Cl}(D)$ is torsion [3, Theorem 3.4]). (Recall that D is a weakly Krull domain if $t\text{-}\dim(D) = 1$ and the intersection $D = \bigcap_{P\in t\text{-}\mathrm{Max}(D)} D_P$ has finite character.)

The Mori–Nagata theorem states that the (complete) integral closure of a Noetherian domain is a Krull domain [26, Theorem 33.10]. We next show that this holds for SM-domains.

Theorem 3.1 (cf. [30, Theorem 3.5]). *If D is an SM-domain, then $D^w = \tilde{D} = D^*$ and D^* is a Krull domain. In particular, if D is integrally closed, then D is a Krull domain.*

Proof. Let $N_v = N_v(D)$, and recall that $\bar{D}[X]_{N_v} \cap \mathrm{qf}(D) = D^w$ (Theorem 1.3), $D[X]_{N_v}$ is a Noetherian domain [12, Theorem 2.2], and $\bar{D}[X]_{N_v}$ is the integral closure (hence complete integral closure) of $D[X]_{N_v}$. If $x \in \mathrm{qf}(D)$ is almost integral over D, then x is almost integral over $D[X]_{N_v}$; so $x \in \bar{D}[X]_{N_v} \cap \mathrm{qf}(D) = D^w$. Hence $D^* \subseteq D^w$, and thus $D^w = \tilde{D} = D^*$ because $D^w \subseteq \tilde{D} \subseteq D^*$. Moreover, since $\bar{D}[X]_{N_v}$ is a Krull domain [26, Theorem 33.10], $D^w = \bar{D}[X]_{N_v} \cap \mathrm{qf}(D)$ is a Krull domain [16, Proposition 1.2]. The "in particular" statement follows because $D = \bar{D} = (\bar{D})^w = D^w$ (Corollary 1.4(1)). $\square$

It is well known that a Noetherian domain D is a UMT-domain if and only if t-$\dim(D) = 1$ [19, Theorem 3.7]. We next generalize this result to SM-domains.

Corollary 3.2. *Let D be an SM-domain which is not a field. Then the following statements are equivalent.*

(1) *D is a UMT-domain.*
(2) *The pair D, D^* satisfies* (#).
(3) t-$\mathrm{Max}(D^*) = \{Q \in \mathrm{Spec}(D^*) \mid Q \cap D \in t\text{-}\mathrm{Max}(D)\}$.
(4) $D^*[X]_{N_v(D)} = D^*[X]_{N_v(D^*)}$.
(5) t-$\mathrm{Max}(D^*) = w_D$-$\mathrm{Max}(D^*)$.
(6) t-$\dim(D) = 1$.
(7) *Each t-linked overring of D is an SM-domain.*
(8) *Each overring of $D[X]_{N_v(D)}$ is a Noetherian domain.*

Proof. Note that $D^w = D^*$ (Theorem 3.1) since D is an SM-domain.

(1) $\Leftrightarrow$ (2). This follows directly from Theorem 2.6 because D^* is a Krull domain (hence a PVMD) (Theorem 3.1).

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). See Lemma 2.4 and Remark 2.5(2).

(2) $\Rightarrow$ (6). Let P be a maximal t-ideal of D, and let Q be a prime ideal of D^* such that $Q \cap D = P$ and $\mathrm{ht}\, Q = \mathrm{ht}\, P$ (cf. Corollary 1.4(3)). By (2), $Q_t \subsetneq D^*$, and hence $\mathrm{ht}\, Q = 1$ because D^* is a Krull domain (Theorem 3.1). Thus $\mathrm{ht}\, P = 1$.

(6) $\Rightarrow$ (1). Recall that $D[X]_{N_v(D)}$ is a one-dimensional Noetherian domain [12, Theorem 2.2 and Corollary 2.4] and $\mathrm{Max}(D[X]_{N_v(D)}) = \{P[X]_{N_v(D)} \mid P \in t\text{-}\mathrm{Max}(D)\}$ [21, Proposition 2.1]. Hence every prime ideal of $D[X]_{N_v(D)}$ is extended from D, and thus D is a UMT-domain [19, Theorem 3.1].

(6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8). See [12, Corollary 3.5]. $\square$

Recall from [24, Theorem] that if D is a weakly factorial SM-domain, then each t-linked overring of D is a weakly factorial SM-domain. In [6, Theorem 3.5], we showed that if D is an almost weakly factorial Noetherian domain, then each integrally closed t-linked overring of D is almost factorial.

Theorem 3.3 (cf. [6, Theorem 3.5] for Noetherian domains). *Let D be an almost weakly factorial SM-domain. Then every t-linked overring of D is an almost weakly factorial SM-domain.*

Proof. Let R be a t-linked overring of D. Then R is an SM-domain with $t\text{-}\dim(R) = 1$ ([24, Lemma 2] or Corollary 3.2) since $t\text{-}\dim(D) = 1$. So we need only show that $\mathrm{Cl}(D)$ is torsion [3, Theorem 3.4]. We shall complete the proof by showing that if I is a t-invertible integral t-ideal of R, then there is a positive integer $n = n(I)$ such that $(I^n)_t$ is principal.

Let $X^1(D)$ and $X^1(R)$ be the sets of height-one prime ideals of D and R, respectively. Let $\Lambda = \{P \in X^1(D) \mid P = Q \cap D$ for some $Q \in X^1(R)$ with $I \subseteq Q\}$ and $\Lambda_1 = \{Q \in X^1(R) \mid Q \cap D \in \Lambda\}$. Since R is an SM-domain with $t\text{-}\dim(R) = 1$, the number of height-one prime ideals of R containing I is finite. Hence Λ, and thus Λ_1, is finite. So if we set $S = R \setminus (\bigcup_{Q \in \Lambda_1} Q)$, then R_S is a semilocal Noetherian domain with $\dim(R_S) = 1$ (cf. [17, Proposition 4.8]). Note that since I is t-invertible, IR_S is invertible, and so $IR_S = xR_S$ for some $x \in R$ [17, Proposition 7.4]. Hence $I = \bigcap_{Q \in X^1(R)} IR_Q = R \cap IR_S = R \cap xR_S$ [21, Proposition 2.8(3)]. Now, since D is an AWFD, there is a positive integer $n = n(x)$ such that $x^n = a/b$, where $a, b \in D$ are products of primary elements of D. Since $a/b \in R_S$, we may assume that $a, b \notin P$ for all $P \in X^1(D) \setminus \Lambda$ (and hence $a, b \notin Q$ for all $Q \in X^1(R) \setminus \Lambda_1$). Also, since $(I^n)_t R_S = ((I^n)_t R_S)_t = (I^n R_S)_t$ ([10, Lemma 2.5] and [21, Lemma 3.4]), we have

$$\left(I^n\right)_t = R \cap \left(I^n\right)_t R_S = R \cap \left((IR_S)^n\right)_t = R \cap x^n R_S = \frac{a}{b} R. \qquad \square$$

The proof of Theorem 3.3 gives another proof of [24, Theorem].

Corollary 3.4 [24, Theorem]. *Every t-linked overring of a weakly factorial SM-domain is a weakly factorial SM-domain.*

Corollary 3.5. *Let D be a weakly* (*respectively, an almost weakly*) *factorial SM-domain. Then each integrally closed t-linked overring of D is a factorial* (*respectively, an almost factorial*) *domain. In particular, D^* is a factorial* (*respectively, an almost factorial*) *domain.*

Proof. Let R be an integrally closed t-linked overring of D. Then R is an integrally closed SM-domain (Corollary 3.2), and hence R is a Krull domain (Theorem 3.1). Note that R is a WFD by Corollary 3.4 (respectively, AWFD by Theorem 3.3); so $\mathrm{Cl}(R) = 0$ [4, Theorem] (respectively, $\mathrm{Cl}(R)$ is torsion [3, Theorem 3.4]). Thus R is factorial (respectively, almost factorial). The "in particular" statement follows because D^* is a Krull domain, $D^w = D^*$ (Theorem 3.1), and D^w is t-linked over D (Lemma 1.2). $\square$

Corollary 3.6. *Let D be an SM-domain with* $t\text{-}\dim(D) = 1$, *and let R be a t-linked overring of D.*

(1) *If* $\mathrm{Cl}(D) = 0$, *then* $\mathrm{Cl}(R) = 0$.
(2) *If* $\mathrm{Cl}(D)$ *is torsion, then* $\mathrm{Cl}(R)$ *is torsion.*

Proof. Recall that if A is an SM-domain with t-$\dim(A) = 1$, then A is a weakly Krull domain; hence A is a weakly (respectively, an almost weakly) factorial domain if and only if $\mathrm{Cl}(A) = 0$ (respectively, $\mathrm{Cl}(A)$ is torsion). Thus the results follow directly from Theorem 3.3 and Corollary 3.4. □

Corollary 3.7. *Let D be a Noetherian domain with* $\dim(D) = 1$. *Then the following statements are equivalent.*

(1) *D is a weakly* (*respectively, an almost weakly*) *factorial domain.*
(2) *Each overring of D is a weakly* (*respectively, an almost weakly*) *factorial domain.*
(3) *If R is an overring of D, then* $\mathrm{Cl}(R) = 0$ (*respectively,* $\mathrm{Cl}(R)$ *is torsion*).

Proof. Note that each overring of D is t-linked over D since $\dim(D) = 1$ and that a Noetherian domain is an SM-domain; so (1) ⇒ (2) is an immediate consequence of Corollary 3.4 (respectively, Theorem 3.3). (2) ⇒ (1) is clear because D is an overring of D itself. (2) ⇔ (3) follows from [4, Theorem] (respectively, [3, Theorem 3.4]). □

Let $N_v = N_v(D)$. Then $\mathrm{Max}(D[X]_{N_v}) = \{P[X]_{N_v} \mid P \in t\text{-}\mathrm{Max}(D)\}$ [21, Proposition 2.1] and $\mathrm{Cl}(D[X]_{N_v}) = 0$ [21, Theorems 2.4 and 2.14]. So $D[X]_{N_v}$ is a WFD if and only if D is a weakly Krull UMT-domain. Moreover, D is an SM-domain with t-$\dim(D) = 1$ if and only if $D[X]_{N_v}$ is a one-dimensional weakly factorial Noetherian domain [12, Theorem 2.2 and Corollary 2.4].

References

[1] D.D. Anderson, S.J. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000) 2461–2475.
[2] D.D. Anderson, E. Houston, M. Zafrullah, t-linked extension, the t-class group, and Nagata's theorem, J. Pure Appl. Algebra 86 (1993) 109–124.
[3] D.D. Anderson, J. Mott, M. Zafrullah, Finite character representations for integral domains, Boll. Unione Mat. Ital. (7) 6-B (1992) 613–630.
[4] D.D. Anderson, M. Zafrullah, Weakly factorial domains and group of divisibility, Proc. Amer. Math. Soc. 109 (1990) 907–913.
[5] D.D. Anderson, M. Zafrullah, Almost Bezout domains, J. Algebra 142 (1991) 285–309.
[6] D.F. Anderson, G.W. Chang, J. Park, Generalized weakly factorial domains, Houston J. Math. 29 (2003) 1–13.
[7] D.F. Anderson, E. Houston, M. Zafrullah, Pseudo-integrality, Canad. Math. Bull. 34 (1991) 15–22.
[8] D.F. Anderson, A. Ryckaert, The class group of $D + M$, J. Pure Appl. Algebra 52 (1988) 199–212.
[9] E. Bastida, R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$, Michigan Math. J. 20 (1973) 79–95.
[10] A. Bouvier, M. Zafrullah, On some class group of an integral domain, Bull. Soc. Math. Greece 29 (1988) 45–59.
[11] J. Brewer, E. Rutter, $D + M$ constructions with general overrings, Michigan Math. J. 23 (1976) 33–42.
[12] G.W. Chang, Strong Mori domains and the ring $D[X]_{N_v}$, J. Pure Appl. Algebra 197 (2005) 293–304.
[13] D. Dobbs, E. Houston, T. Lucas, M. Roitman, M. Zafrullah, On t-linked overrings, Comm. Algebra 20 (1992) 1463–1488.
[14] D. Dobbs, E. Houston, T. Lucas, M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17 (1989) 2835–2852.

[15] M. Fontana, S. Gabelli, E. Houston, UMT-domains and domains with Prüfer integral closure, Comm. Algebra 26 (1998) 1017–1039.
[16] R. Fossum, The Divisor Class Group of a Krull Domain, Springer, New York, 1973.
[17] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[18] J. Hedstrom, E. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44.
[19] E. Houston, M. Zafrullah, On t-invertibility, II, Comm. Algebra 17 (1989) 1955–1969.
[20] H.C. Hutchins, Examples of Commutative Rings, Polygonal Publishing House, Passaic, NJ, 1981.
[21] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989) 151–170.
[22] I. Kaplansky, Commutative Rings, revised ed., Univ. of Chicago, Chicago, 1974.
[23] H. Kim, Y.S. Park, Some remarks on pseudo-Krull domains, preprint.
[24] M. Martin, M. Zafrullah, t-linked overrings of Noetherian weakly factorial domains, Proc. Amer. Math. Soc. 115 (1992) 601–604.
[25] J. Mott, M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981) 1–26.
[26] M. Nagata, Local Rings, Interscience, New York, 1962.
[27] U. Storch, Fastfaktorielle Ringe, Schriftenreihe Math. Inst. Univ. Münster, Heft 36, 1972.
[28] F. Wang, w-dimension of domains, Comm. Algebra 27 (1999) 2267–2276.
[29] F. Wang, On induced operations and UMT-domains, Sichuan Shifan Daxue Xuebao Ziran Kexue Ban 27 (2004) 1–9.
[30] F. Wang, R.L. McCasland, On strong Mori domains, J. Pure Appl. Algebra 135 (1999) 155–165.